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SPECTRA AND MEASURE INEQUALITIES(1)

BY

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ABSTRACT. Let T be a bounded operator on a Hilbert space $\mathfrak{H}$ and let $T_z = T - zI$. Then the operators $T_z T_z^*$, $T_z T_t (T_z T_t)^*$, and $T_z T_t T_s (T_z T_t T_s)^*$ are nonnegative for all complex numbers z, t, and s. We shall obtain some norm estimates for nonnegative lower bounds of these operators, when z, t, and s are restricted to certain sets, in terms of certain capacities or area measures involving the spectrum and point spectrum of T. A typical such estimate is the following special case of Theorem 4 below: Let $\mathfrak{H}$ be separable and suppose that $T_z T_t (T_z T_t)^* \geqslant D \geqslant 0$ for all z and t not belonging to the closure of the interior of the point spectrum of T. In addition, suppose that the boundary of the interior of the point spectrum of T has Lebesgue planar measure 0. Then $\|D\|^{1/2} \leqslant \pi^{-1} \operatorname{meas}_2(\sigma_p(T))$. If T is the adjoint of the simple unilateral shift, then equality holds with $D = I - T^*T$.

1. Introduction. Let T be an operator (bounded, linear transformation) on a Hilbert space $\mathfrak{H}$ with spectrum $\sigma(T)$ and point spectrum $\sigma_p(T)$. If z is any complex number let $T_z = T - zI$ and, for a fixed positive integer n and arbitrary complex numbers $z_1, \ldots, z_n$, let

$$T(z_1, \ldots, z_n) = (T_{z_1} \cdots T_{z_n})(T_{z_1} \cdots T_{z_n})^*.$$

Obviously, $T(z_1, \ldots, z_n)$ is a nonnegative operator invariant under any permutation of the n variables $z_1, \ldots, z_n$. For a fixed n and a fixed subset S of the complex plane let D be any operator satisfying

$$T(z_1, \ldots, z_n) \geqslant D \geqslant 0 \quad \text{for all } z_1, \ldots, z_n \notin S. \tag{1.1}$$

(Trivially, $D = 0$ satisfies (1.1) for any n and S.) We shall be concerned below with some inequalities of the type

$$\|D\| \leqslant k\mu(S) \qquad (k = \text{const}), \tag{1.2}$$

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particularly in the cases $n = 1, 2, 3$, where $\mu(S)$ denotes a certain capacity or measure of S.

Recall that T is said to be cohyponormal if T^* is hyponormal, so that $TT^* - T^*T = D \geqslant 0$. Since $TT^* - T^*T$ is unchanged if T is replaced by $T_z = T - zI$, it is seen that if T is cohyponormal then

(1.3) $$T_z T_z^* \geqslant D = TT^* - T^*T, \quad \text{for all } z \in \mathbf{C}.$$

In this case it is known (Putnam [8]) that

(1.4) $$\|D\| \leqslant \pi^{-1} \operatorname{meas}_2 \sigma(T).$$

Since the inequality of (1.3) holds, *a fortiori*, for $z \notin \sigma(T)$, it is seen that (1.4) is an instance of (1.2) where $n = 1$, $s = \sigma(T)$, and μ is Lebesgue planar measure.

In case S is a sufficiently large set, for instance, if S is any open set containing $\sigma(T)$, where T is now an arbitrary operator, then $T(z_1, \ldots, z_n) \geqslant c_n I$ ($c_n = \text{const} > 0$), where $z_1, \ldots, z_n \notin S$. On the other hand, if S is too small, for instance, if S is the empty set, it may turn out that the only operator D satisfying relation (1.1) (n fixed) is $D = 0$. Thus, if T is normal, the only $D \geqslant 0$ satisfying $T_z T_z^* \geqslant D$ for all z is $D = 0$; cf. Putnam [10, p. 168]. (This assertion follows from the fact that if T is normal then $x = 0$ is the only element in the range of T_z for all $z \in \mathbf{C}$. This last result was obtained in [9] but was given earlier by Johnson [6], in a paper which came to our attention after the appearance of [9].) However, for any operator T, if $T(z_1, \ldots, z_n) \geqslant D \geqslant 0$ (for a fixed n and $z_1, \ldots, z_n$ arbitrary), then $T_z T_z^* \geqslant cD$ (z arbitrary) for some $c = \text{const} > 0$, as can be seen by letting $z_1 = z$ and by choosing $z_2, \ldots, z_n$ to be any fixed numbers lying outside $\sigma(T)$. Consequently, if n is fixed and if T is normal, the only D satisfying $T(z_1, \ldots, z_n) \geqslant D \geqslant 0$ for all $z_1, \ldots, z_n$ in $\mathbf{C}$ is $D = 0$.

It turns out that if Q denotes the adjoint of the simple unilateral shift then

(1.5) $$Q(z_1, \ldots, z_n) \geqslant c_n D, \quad \text{for all } z \in \mathbf{C},$$

where $D = I - Q^*Q$ ($\neq 0$) and $c_n = \text{const} > 0$ ($n = 1, 2, \ldots$). In fact, for $n = 1$, one can choose $c_n = 1$, as was noted above. If $n = 2$, and if $z_1 = z$ and $z_2 = t$, then

$$Q(z,t) = Q_z Q_t (Q_z Q_t)^* = Q_z (I + |t|^2 I - tQ^* - \bar{t}Q) Q_z^*.$$

Since $\|Q\| = 1$, this implies that $Q(z,t) \geqslant (1 - |t|)^2 Q_z Q_z^* \geqslant (1 - |t|)^2 D$. Also $Q(z,t) \geqslant Q_z D Q_z^* = |z|^2 D$, the last equality by direct calculation. Since each of the above relations holds with z and t interchanged, it follows that

$$Q(z,t) \geqslant \tfrac{1}{4}(|z|^2 + (1-|z|)^2 + |t|^2 + (1-|t|)^2)D \geqslant \tfrac{1}{4}(\tfrac{1}{2}+\tfrac{1}{2})D = \tfrac{1}{4}D,$$

so that one may take $c_2 = \frac{1}{4}$ in (1.5). Similar arguments can be used to establish (1.5) in general, and we omit the details.

The analytic capacity and the continuous analytic capacity of a set S will be denoted by $\gamma(S)$ and $\alpha(S)$, respectively. Excellent sources for the definitions and properties of these capacities are Gamelin [4], Garnett [5], and Zalcman [13]. We recall a few properties. Let S^2 denote the Riemann sphere and let Ω be any open subset of the plane. Let $H^\infty(\Omega)$ denote the bounded analytic functions on Ω and $A(\Omega)$ those functions in $H^\infty(\Omega)$ having continuous extensions to S^2. Let E be a compact set of $\mathbf{C}$ and let $f(z)$ be analytic on $\Omega = S^2 - E$. Then (cf. [5, p. 6]),

$$f'(\infty) = \frac{1}{2\pi i}\int_C f(z)\,dz,$$

where C is a positively oriented rectifiable simple closed curve containing E in its interior, and

$$\gamma(E) = \sup\{|f'(\infty)| : f \in H^\infty(\Omega),\ |f(z)| \leqslant 1 \text{ on } \Omega\}, \tag{1.6}$$

and

$$\alpha(E) = \sup\{|f'(\infty)| : f \in A(\Omega),\ |f(z)| \leqslant 1 \text{ on } S^2\}. \tag{1.7}$$

For an arbitrary set F one puts $\gamma(F) = \sup\{\gamma(E): E \text{ compact}, E \subset F\}$ and $\alpha(F) = \sup\{\alpha(E): E \text{ compact}, E \subset F\}$. If E is any Borel set of $\mathbf{C}$ then

$$(\operatorname{meas}_2 E/\pi)^{1/2} \leqslant \alpha(E) \leqslant \gamma(E) \leqslant \operatorname{diam} E; \tag{1.8}$$

see [5, pp. 9, 79]. In particular, if E is the unit disk $|z| \leqslant 1$, the first two inequalities of (1.8) are equalities.

It may be noted that those compact sets E for which $\gamma(E) = 0$ are precisely the removable compact sets for bounded analytic functions. Further, if E is a compact subset of an open disk δ and if $\alpha(E) = 0$ then any function continuous on δ and analytic on $\delta - E$ is necessarily analytic on δ. Cf. also the remarks in Putnam [10, p. 169].

In the sequel, if S is any subset of $\mathbf{C}$, we shall denote its closure, interior, and complement by S^-, $\operatorname{int} S$, and S^c respectively.

2. **Case** $n = 1$. Here we consider (1.1) for $n = 1$ and $S = \varnothing$ and obtain certain estimates for the norm of D.

THEOREM 1. *Suppose that*

$$T_z T_z^* \geqslant D \geqslant 0 \quad \textit{for all } z \in U, \tag{2.1}$$

where U denotes the unbounded component of the complement of $\sigma(T)$. Then

$$\|D\|^{1/2} \leqslant \gamma(\sigma(T)). \tag{2.2}$$

PROOF. Let D have the spectral resolution $D = \int_0^\infty u\, dF_u$ and let $x \in \mathfrak{H}$ satisfy $k_x = \int_{+0}^\infty u^{-1} d\|F_u x\|^2 < \infty$. It follows from Putnam [11] that $w(z) = T_z^{-1} x$ on U satisfies

$$\|w(z)\| \leqslant k_x^{1/2}. \tag{2.3}$$

On $\mathbf{C} - \sigma(T)$, $w(z)$ is, of course, analytic and $w(z) \to 0$ (strongly) as $|z| \to \infty$.

Let C be a positively oriented rectifiable simple closed curve containing $E = \sigma(T)$ in its interior, and let $f(z) = (w(z), x)$, so that $f(z)$ is analytic and bounded on U. Since $I = -(2\pi)^{-1} \int_C T_z^{-1}\, dz$, it follows from (1.6) that

$$\|x\|^2 = -\frac{1}{2\pi i}\int_C f(z)\, dz \leqslant \gamma(\sigma(T)) \sup_{z \in U} |f(z)| \leqslant \gamma(\sigma(T)) k_x^{1/2} \|x\|.$$

On choosing a sequence $x = x_n$ of unit vectors so that

$$k_{x_n} = \int_{+0}^\infty u^{-1} d\|F_u x_n\|^2 \to \|D\|^{-1},$$

one obtains (2.2).

A result related to Theorem 1 is

THEOREM 2. *Suppose that $T_z T_z^* \geqslant D \geqslant 0$ for all z in $\mathbf{C}$. Let*

$$K = [\sigma_p(T) \cap \partial\sigma(T)]^-$$

and suppose that

$$\gamma(K) = 0 \tag{2.4}$$

and that N is an open set satisfying

$$K \subset N \quad \text{and} \quad \alpha(N \cap \sigma(T)) = 0. \tag{2.5}$$

(*Hence, in particular, no interior points of $\sigma(T)$ lie in N.*) *Then*

$$\|D\|^{1/2} \leqslant \alpha(\sigma(T)). \tag{2.6}$$

PROOF. Let $z_0 \in N - K$ and let $x \in \mathfrak{H}$ and $w(z)$ be defined as in the proof of Theorem 1. It follows from [11] that $w(z)$ has an extension to the entire plane which satisfies (2.3) on $\mathbf{C}$ and which is (weakly) continuous on $\mathbf{C} - \sigma_p(T)$. Then there exists an open disk δ about z_0 for which $\delta \cap \sigma_p(T)$

$= \varnothing$, and, hence, $f(z) = (w(z), x)$ is continuous on δ and analytic on $\delta - \sigma(T)$. Since $\alpha(N \cap \sigma(T)) = 0$, then $f(z)$ is analytic on δ, and so $f(z)$ is analytic and bounded on $N - K$. Since $\gamma(K) = 0$, then $f(z)$ has an analytic extension in N, and therefore an extension which is continuous on K and hence also on $[(\sigma(T))^c]^-$. On $\operatorname{int}\sigma(T)$, one can, if necessary, redefine $f(z)$ so as to obtain a function $f^*(z)$ which is continuous on S^2 and coincides with $f(z)$ off $\sigma(T)$, and which satisfies $\sup_{S^2}|f^*(z)| \leqslant \sup_{\mathbf{C}}|f(z)|$. An argument similar to that used in establishing Theorem 1 then implies (2.6).

COROLLARY OF THEOREM 2. *If $T_z T_z^* \geqslant D \geqslant 0$ for all z in $\mathbf{C}$ and if $\sigma_p(T) \cap \partial(\sigma(T)) = \varnothing$ (in particular, if $\sigma_p(T) = \varnothing$), then $\|D\|^{1/2} \leqslant \alpha(\sigma(T))$.*

In case $T = Q$, where Q^* is the simple shift, then $\sigma(T)$ is the closed unit disk and $\sigma_p(T)$ is the open disk, so that K of Theorem 2 is empty. Also $T_z T_z^* \geqslant D \geqslant 0$ holds for all $z \in \mathbf{C}$ with $D = TT^* - T^*T (= I - T^*T)$. In addition, $\|D\| = 1$, so that both (2.2) and (2.6) become equalities.

Theorem 2 generalizes Corollary 2 of Theorem 4 in Putnam [10]. (The set P occurring there should have been $P = \{z\colon z \text{ in } \sigma_p(T) \text{ or } \bar{z} \text{ in } \sigma_p(T^*)\}$.)

3. **Case** $n = 2$. In this section we shall prove two theorems:

THEOREM 3. *Let T be an operator on a Hilbert space satisfying*

$$T_z T_t (T_z T_t)^* \geqslant D \geqslant 0 \quad \text{for all } z, t \notin \sigma(T). \tag{3.1}$$

Then

$$\|D\|^{1/2} \leqslant \pi^{-1} \operatorname{meas}_2 \sigma(T). \tag{3.2}$$

THEOREM 4. *Let T be an operator on a separable Hilbert space satisfying*

$$T_z T_t (T_z T_t)^* \geqslant D \geqslant 0 \quad \text{for all } z, t \notin E = \sigma_p(T) \cup (\operatorname{int} \sigma(T))^-. \tag{3.3}$$

Then

$$\|D\|^{1/2} \leqslant \pi^{-1} \operatorname{meas}_2 E. \tag{3.4}$$

The separability of $\mathfrak{H}$ in Theorem 4 ensures the planar measurability of $\sigma_p(T)$ and hence of the set E. In fact, $\sigma_p(T)$ is in this case even an F_σ; see Dixmier and Foiaş [3], also Nikolskaya [7]. Aside from the separability hypothesis of Theorem 4, the two theorems are still independent. The fact that the estimate on the right of (3.4) is not larger than that on the right of (3.2) is counterbalanced by the fact that any operator D satisfying (3.3) also serves as a D in (3.1).

COROLLARY OF THEOREM 4. *Let T be an operator on a Hilbert space (separable or not) for which $\sigma_p(T)$ has planar measure 0. Then the only selfadjoint operator D satisfying $T_zT_t(T_zT_t)^* \geqslant D \geqslant 0$ for all $z, t \in \mathbf{C}$ is $D = 0$.*

As we have noted in §1, if $T = Q$, where Q^* is the simple shift, then $T_zT_t(T_zT_t)^* \geqslant D \geqslant 0$ holds for all z, t in $\mathbf{C}$ with $D = \frac{1}{4}(I - T^*T) \neq 0$. In this case, of course, $\sigma_p(T)$ is the open unit disk. When $T = Q^*$ the operators D in both (3.1) and (3.3) can be chosen to be $I - T^*T$. This can be seen from §1 where it was noted that $T_zT_t(T_zT_t)^* \geqslant |z|^2(I - T^*T)$ for arbitrary $z, t \in \mathbf{C}$. Thus, in particular, both estimates (3.2) and (3.4) are sharp.

In view of continuity considerations, it is clear that (3.3) holds if and only if $T_zT_t(T_zT_t)^* \geqslant D \geqslant 0$ holds for all $z, t \notin (\operatorname{int}\sigma_p(T))^-$. In addition, one easily obtains the special case of Theorem 4 stated in the Abstract.

Before giving the proofs of Theorems 3 and 4, we note that one need not resort to, say, $T = Q^*$ in order for equality to obtain in (3.2). For instance, let T be normal with the spectral resolution $T = \int u\,dE_u$ and suppose that $\sigma(T) = \{z: |z| \leqslant 1\}$ and that $0 \in \sigma_p(T)$. Then, for $z, t \notin \sigma(T)$, $T_zT_t(T_zT_t)^* \geqslant E(\{0\})$. Obviously, equality holds in (3.2) with $D = E(\{0\})$.

PROOF OF THEOREM 3. Let D of (3.1) have the spectral resolution $D = \int u\,dF_u$ and let $x \in \mathfrak{H}$ satisfy $k_x = \int_{+0}^{\infty} u^{-1}\,d\|F_ux\|^2 < \infty$. By an argument similar to that given in [11] (summarized briefly at the beginning of §2 above) one can show that there exists a vector-valued function $w(z,t) = w(t,z)$ defined for $z, t \notin \sigma(T)$ such that

$$T_zT_tw(z,t) = x \quad \text{and} \quad \|w(z,t)\| \leqslant k_x^{1/2}. \tag{3.5}$$

Since $z, t \notin \sigma(T)$, then $w(z,t) = (T_zT_t)^{-1}x$.

Next, for $z \notin \sigma(T)$, let $w(z) = T_zw(z,z)$, so that $T_zw(z) = x$ and $T_tw(t) = x$. Hence $T_tT_zw(z) = T_tx$ and $T_zT_tw(t) = T_zx$ and so $T_zT_t(w(z) - w(t)) = (z - t)x$. Consequently, by (3.5),

$$w(z) - w(t) = (z - t)w(z,t) \quad \text{if } z, t \notin \sigma(T). \tag{3.6}$$

For any $\varepsilon > 0$, there exists a bounded open set $\alpha \supset \sigma(T)$ such that $\operatorname{meas}_2(\alpha - \sigma(T)) < \varepsilon$ and $\partial\alpha$ consists of a finite number of nonintersecting polygons lying in the resolvent set of T and having horizontal and vertical sides. To see this, first cover $\sigma(T)$ with a (finite or infinite) sequence of open rectangles $R_1, R_2, \ldots,$ with sides parallel to the coordinate axes and such that $\operatorname{meas}_2(\bigcup R_n - \sigma(T)) < \varepsilon$. An application of the Borel covering theorem then yields the desired polygons forming the boundary of α, and which we suppose are oriented positively with respect to α.

Next, let L be any vertical line $\operatorname{Re}(z) = c$ intersecting α. Then $L \cap \alpha^-$ is a finite union of disjoint closed intervals $J_1, \ldots, J_p$ where $J_k = \{c + iy: a_k \leqslant y$

$\leqslant b_k\}$. Further, $\int_{\partial\alpha} w(z)\,dz = \int_H w(z)\,dz + \int_V w(z)\,dz$, where H and V denote, respectively, the oriented horizontal and vertical segments of $\partial\alpha$. Then, by (3.6), the contribution to $\int_H w(z)\,dz$ of $w(z)\,dz$ for $\operatorname{Re}(z) = c$ ($z = x + iy$) is a sum

$$-\sum_{k=1}^{p} (w(c + ib_k) - w(c + ia_k))\,dx = -\sum_{k=1}^{p} (b_k - a_k) w(c + ib_k, c + ia_k)\,dx$$

(where $dx \geqslant 0$). In view of (3.5), this last sum is in norm majorized by

$$\sum (b_k - a_k) k_x^{1/2}\,dx = q(c) k_x^{1/2}\,dx,$$

where $q(c)$ denotes the linear measure of the cross-section $L \cap \alpha^-$. Consequently,

$$\left\| \int_H w(z)\,dz \right\| \leqslant k_x^{1/2} \int q(x)\,dx = k_x^{1/2} \operatorname{meas}_2 \alpha.$$

After a similar argument applied to $\int_V w(z)\,dz$, one has

$$\left\| \int_{\partial\alpha} w(z)\,dz \right\| \leqslant \left\| \int_H w(z)\,dz \right\| + \left\| \int_V w(z)\,dz \right\| \leqslant 2k_x^{1/2} \operatorname{meas}_2 \alpha.$$

Since $z \notin \sigma(T)$, $w(z) = T_z^{-1} x$, and so

$$\begin{aligned} \|x\| &= \left\| -(2\pi i)^{-1} \int_{\partial\alpha} w(z)\,dz \right\| \leqslant (2\pi)^{-1} (2k_x^{1/2}) \operatorname{meas}_2 \alpha \\ &\leqslant \pi^{-1} k_x^{1/2} (\operatorname{meas}_2 \sigma(T) + \varepsilon). \end{aligned}$$

By choosing unit vectors $x = x_n$ so that $k_{x_n} \to \|D\|^{-1}$ and noting that $\varepsilon > 0$ is arbitrary, we obtain (3.2), and the proof of Theorem 3 is complete.

PROOF OF THEOREM 4. In view of (3.3), an argument similar to that at the beginning of the proof of Theorem 3 shows that (3.5) again holds, but where now $z, t \notin E = \sigma_p(T) \cup (\operatorname{int} \sigma_p(T))^-$. Corresponding to (3.6) we now have

$$w(z) - w(t) = (z - t) w(z, t), \tag{3.7}$$

for $z, t \notin E$ (hence $z, t \notin \sigma_p(T)$), where $w(z) = T_z w(z,z)$ is defined for $z \notin E$. Also, $w(z,t) = w(t,z)$ and, since $z, t \notin \sigma_p(T)$, $w(z,t)$ is continuous on $E^c \times E^c$.

If $z \in \sigma_p(T) - (\operatorname{int} \sigma_p(T))^-$, let $\{z_n\}$ be any sequence such that $z_n \notin E$ and $z_n \to z$. Then $w(z_n) \to$ limit (strongly). In fact, if $\{z'_n\}$ is another sequence such that $z'_n \notin E$, $z'_n \to z$, then, by (3.7), $w(z_n) - w(z'_n) = (z_n - z'_n) w(z_n, z'_n) \to 0$, since $w(z_n, z'_n)$ is bounded. Thus, the domain of definition of $w(z)$ can be extended to the set $F = \mathbf{C} - (\operatorname{int} \sigma_p(T))^-$, so that, on this set, the extended function, which will also be denoted by $w(z)$, is continuous. Also, if the domain of definition of $w(z,t)$ is extended via relation (3.7) from $E^c \times E^c$ to include also those points (z,t) of $F \times F$ where $z \neq t$, then (3.7) holds whenever $z, t \in F$ and $z \neq t$, where the extensions of $w(z,t)$ and $w(z)$ are denoted by the same symbols. As before, $w(z)$ is analytic on $\mathbf{C} - \sigma(T)$.

Let C denote any positively oriented rectifiable simple closed curve such that C together with its interior C_{int} lies in F and put $\mu(C_{\text{int}}) = -(2\pi i)^{-1} \int_C (w(z),x)\,dz$. In case C is a circle of center b and radius r one has, on using (3.7),

$$\mu\{z\colon |z-b| < r\} = -\frac{1}{2\pi i}\int_C (z-b)(w(z,b),x)\,dz$$

and hence, by (3.5),

$$|\mu\{z\colon |z-b| < r\}| \leqslant (2\pi)^{-1} k_x^{1/2} \|x\| (2\pi r^2) = \pi^{-1} k_x^{1/2} \|x\| m\{z\colon |z-b| < r\},$$

where $m = \text{meas}_2$. It follows that μ can be extended to a complex measure on the Borel sets of F and that the resulting measure, which we also denote by μ, is absolutely continuous with respect to Lebesgue planar measure. (In view of (3.7), $f(z) = (w(z),x)$ satisfies a Lipschitz condition on the open set F. The above set function μ on the Borel sets of F may be compared with the set function ν on the Borel sets of $\mathbf{C}$ occurring in Garnett [5, p. 47]. For a general discussion of Borel measures, see, e.g., Rudin [12].) In addition, one sees that the Radon-Nikodym derivative $d\mu/dm$ satisfies $|d\mu/dm| \leqslant \pi^{-1} k_x^{1/2} \|x\|$ on F. Further, if $b \in F - \sigma_p(T)$, then $w(z,b) \to w(b,b)$ (weakly) as $z \to b$. Hence, if $C = \{z\colon |z-b| = r\}$, then

$$\mu\{z\colon |z-b| < r\} = -\frac{1}{2\pi i}\int_C (z-b)(w(z,b) - w(b,b),x)\,dz = o(r^2),$$

as $r \to 0$, so that $d\mu/dm = 0$ on $F - \sigma_p(T)$. Thus (dm-almost everywhere),

$$\text{(3.8)} \quad |d\mu/dm| \leqslant \pi^{-1} k_x^{1/2} \|x\| \text{ on } F \quad \text{and} \quad d\mu/dm = 0 \text{ on } F - \sigma_p(T).$$

It is now easy to complete the proof of Theorem 4. If $\operatorname{int}\sigma_p(T) \neq \varnothing$, let the role of $(\operatorname{int}\sigma_p(T))^-$ correspond to that of $\sigma(T)$ in the beginning of the proof of Theorem 3. Then choose an open set $\alpha \supset (\operatorname{int}\sigma_p(T))^-$ such that

$$\text{meas}\{\alpha - (\operatorname{int}\sigma_p(T))^-\} < \varepsilon$$

with a properly oriented boundary $\partial\alpha$ consisting of a finite number of polygons of horizontal and vertical segments lying in the complement of $(\operatorname{int}\sigma_p(T))^-$. Then

$$(3.9)\qquad \left|-\frac{1}{2\pi i}\int_{\partial\alpha}(w(z),x)\,dz\right| \leqslant \frac{1}{\pi}k_x^{1/2}\|x\|(m(\operatorname{int}\sigma_p(T))^- + \varepsilon).$$

Next, let C denote the positively oriented circle $|z| = R$, where R is so large that $\sigma(T)$ and $\partial\alpha$ belong to the interior of C. Since $w(z) = T_z^{-1}x$ on C we have

$$(3.10)\qquad (x,x) = -\frac{1}{2\pi i}\int_{C-\partial\alpha}(w(z),x)\,dz - \frac{1}{2\pi i}\int_{\partial\alpha}(w(z),x)\,dz.$$

Since the first integral is $J = \int_{\sigma(T)\cap\alpha^c}(d\mu/dm)\,dm$, it follows from (3.8) that

$$|J| = \left|\int_{\sigma_p(T)\cap\alpha^c}\frac{d\mu}{dm}dm\right| \leqslant \pi^{-1}k_x^{1/2}\|x\|m(\sigma_p(T)\cap\alpha^c)$$
$$\leqslant \pi^{-1}k_x^{1/2}\|x\|m[\sigma_p(T) - (\operatorname{int}\sigma_p(T))^-].$$

Since $\varepsilon > 0$ is arbitrary, it follows from (3.9) and (3.10) that

$$\|x\|^2 \leqslant \pi^{-1}k_x^{1/2}\|x\|m[\sigma_p(T)\cup(\operatorname{int}\sigma_p(T))^-].$$

Hence, if $x = x_n$ are unit vectors for which $k_{x_n} \to \|D\|^{-1}$, one obtains (3.4). This completes the proof of Theorem 4.

4. **Case** $n = 3$. We now suppose that

$$(4.1)\quad T_zT_tT_s(T_zT_tT_s)^* \geqslant D \geqslant 0 \quad \text{for } z, t, s \notin E = \sigma_p(T)\cup(\operatorname{int}\sigma_p(T))^-.$$

THEOREM 5. *If T is an operator on a Hilbert space satisfying* (4.1) *then*

$$(4.2)\quad \|D\|^{1/2} \leqslant \pi^{-1}\operatorname{meas}_2(\operatorname{int}\sigma_p(T))^-[\sup\{\|T_z\|: z\in\partial(\operatorname{int}\sigma_p(T))^-\}].$$

PROOF. The proof is similar to those of Theorems 3 and 4 above. If $D = \int u\,dF_u$ and if $x \in \mathfrak{H}$ satisfies $k_x = \int_{+0}^{\infty}u^{-1}d\|F_ux\|^2 < \infty$, then by an argument like that in [11], one can show that there exists a vector-valued function $w(z,t,s)$ defined, and invariant under permutation of its arguments, for $z, t, s \in E^c$ and satisfying

$$(4.3)\qquad T_zT_tT_sw(z,t,s) = x \quad\text{and}\quad \|w(z,t,s)\| \leqslant k_x^{1/2}.$$

Also, $w(z,t,s)$ is continuous on $E^c\times E^c$ since $z, t, s \notin \sigma_p(T)$.

Next, we put $w(z) = T_z^2w(z,z,z)$ and $u(z,t) = T_zw(z,z,t)$, where $z, t \in E^c$. Then $T_zw(z) = x$ and, as in §3,

$$(4.4)\qquad T_zT_t(w(z) - w(t)) = (z-t)x.$$

Similarly, $T_z T_t u(z,t) = T_t T_t u(t,t) = x$ and, hence,

$$T_z T_t T_t(u(z,t) - u(t,t)) = (z-t)x. \tag{4.5}$$

If $z, t \in E^c$, then, by (4.3)–(4.5), we have

$$w(z) = w(t) + (z-t)u(z,t) \tag{4.6}$$

and

$$u(z,t) = u(t,t) + (z-t)w(z,t,t). \tag{4.7}$$

Since $w(z,t,t)$ is bounded (for $z, t \in E^c$), it is clear from (4.6) and (4.7) (cf. the proof of Theorem 4) that $u(z,t)$ has a continuous extension to the set $F \times F$, and $w(z)$ a continuous extension to the set F, where $F = \mathbf{C} - (\operatorname{int}\sigma_p(T))^-$. If the domain of definition of $w(z,t,t)$ is extended from $E^c \times E^c$ via (4.7) to include also those points (z,t) of $F \times F$ for which $z \neq t$, then the extensions (denoted by the same symbols) satisfy (4.6) for $z, t \in F$ and, at least if $z \neq t$, also (4.7). In particular, $w(z)$ is analytic on F.

Let α be chosen as in the proof of Theorem 4. Since $w(z)$ is analytic on F and since $\|u(z,t)\| \leqslant \|T_z w(z,z,t)\|$ holds for $z, t \in E^c$, and even for $z, t \in F$ if $z \neq t$, an argument like that used in the proof of Theorem 4 shows that

$$\|x\|^2 = -\frac{1}{2\pi i}\int_{\partial\alpha} (w(z),x)\,dz \leqslant \frac{1}{\pi}\|x\| M k_x^{1/2}(\text{meas}_2\,(\operatorname{int}\sigma_p(T))^- + \varepsilon),$$

where $M = \sup\{\|T_z\| : z \in \alpha - (\operatorname{int}\sigma_p(T))^-\}$. On choosing a sequence of open sets $\alpha = \alpha_n$ contracting to $(\operatorname{int}\sigma_p(T))^-$ and unit vectors $x = x_n$ for which $k_{x_n} \to \|D\|^{-1}$, one obtains (4.2) and the proof of Theorem 5 is complete.

5. **Remarks.** It was noted in §1 that if T^* is hyponormal then (1.3) holds. In particular, if T^* is completely hyponormal, so that T^* has no normal part, then $T_z T_z \geqslant D \geqslant 0$ holds for some $D \neq 0$. Further, if $T^* = Q$ is the unilateral shift, then (1.5) even holds.

In view of the corollary to Theorem 4, it is seen that if $T_z T_t (T_z T_t)^* \geqslant D \geqslant 0$ holds for all $z, t \in \mathbf{C}$, with $D \neq 0$, then necessarily $\sigma_p(T)$ has positive planar measure. However, as was shown by Clancey and Morrel [2, p. 133], using a result of Brennan [1], there exists completely subnormal operators T^* for which $\sigma_p(T)$ is empty. Thus, for such an operator, $T_z T_t (T_z T_t)^* \geqslant D \geqslant 0$ for all $z, t \in \mathbf{C}$ can hold only for $D = 0$.

It follows from Theorem 5 that if $\sigma_p(T)$ has no interior (in particular, if $\sigma(T)$ has no interior) then $T_z T_t T_s (T_z T_t T_s)^* \geqslant D \geqslant 0$ can hold for all $z, t, s \in \mathbf{C}$ only for $D = 0$.

We do not know whether the inequalities (1.1) for $n \geqslant 4$ imply some sort of results comparable to the assertions of Theorems 1–5 for $n \leqslant 3$. Although, for a fixed $n \geqslant 4$, one could still obtain a vector function $w(z)$ playing a role similar to that of the corresponding function for $n \leqslant 3$, nevertheless, the relation (1.5) suggests that the chain of inequality relations (including (2.2) and (2.6) when $n = 1$, (3.2) and (3.4) when $n = 2$, and (4.2) when $n = 3$) may effectively end at $n = 3$.

ADDED 9/20/76. In connection with the above remarks concerning [1] and [2] the referee has kindly pointed out a recent relevant paper of C. Fernstrøm appearing in Lecture Notes in Math., vol. 512, Springer-Verlag, Berlin and New York.

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